Grade 9/10 Math Circles
November 8, 2023
Graph Theory - Solutions

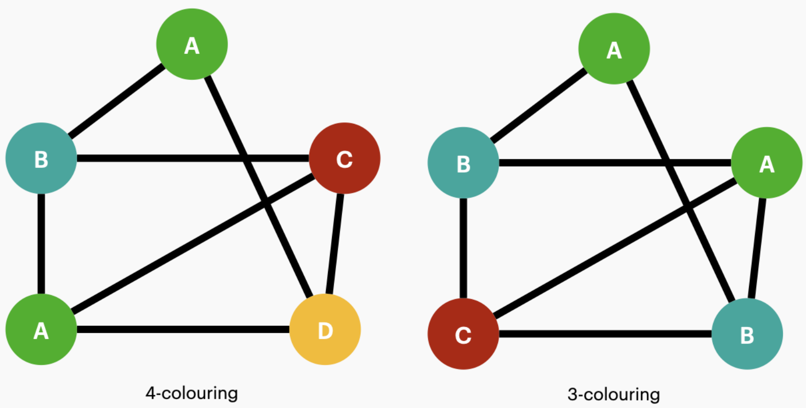
Exercise Solutions

Exercise 1

Draw a graph then swap pages with someone else and give the graph a colouring!

Exercise 1 Solution

Answers may vary! Here are a couple examples:



Exercise 2

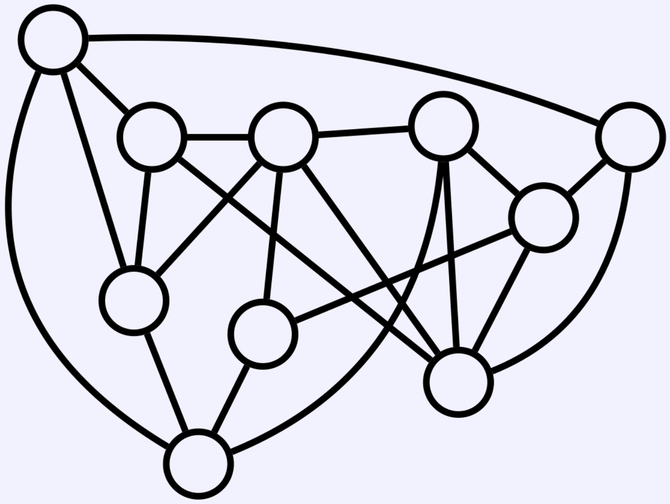
When is a graph 1-colourable? When is a graph 2-colourable?

Exercise 2 Solution

A graph is 1-colourable if it has no edges. A graph is 2-colourable if we have no cycles of odd length.

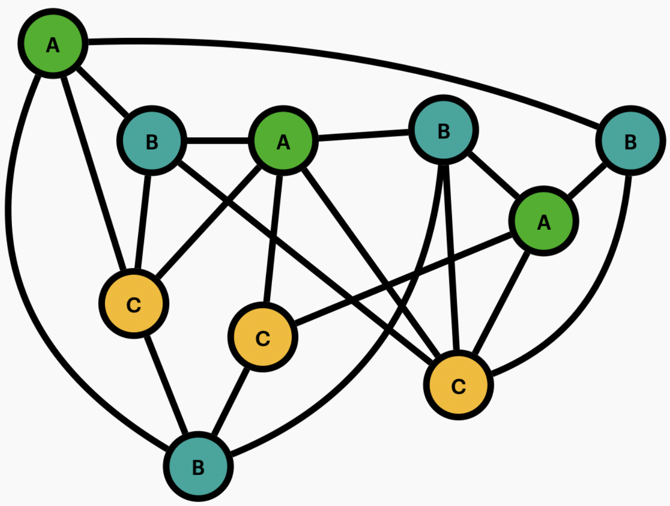
Exercise 3

Colour the following graph using the greedy colouring algorithm.



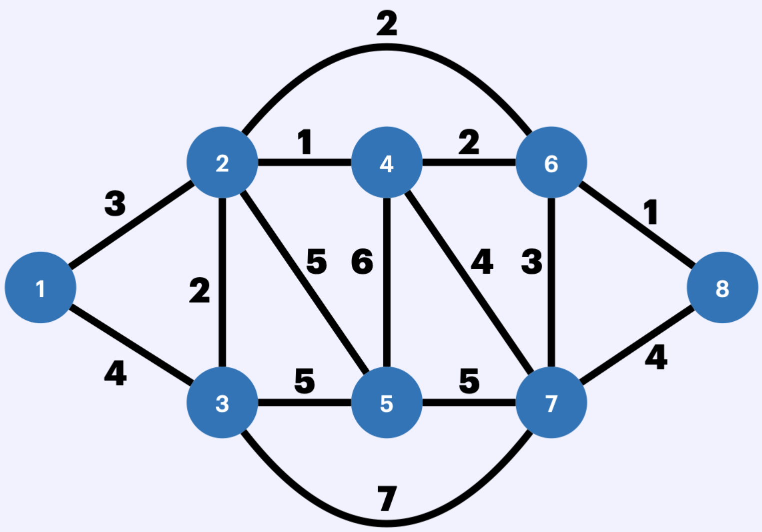
Exercise 3 Solution

Below is an example result:



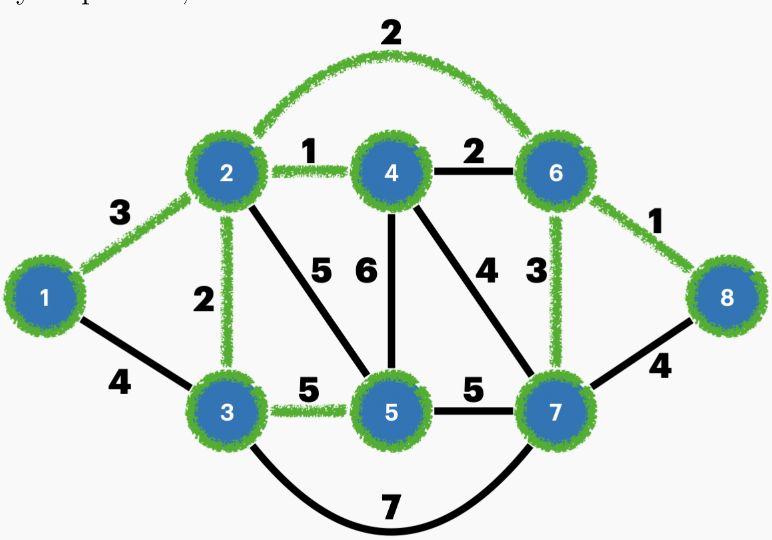
Exercise 4

Find an MST of the graph below:



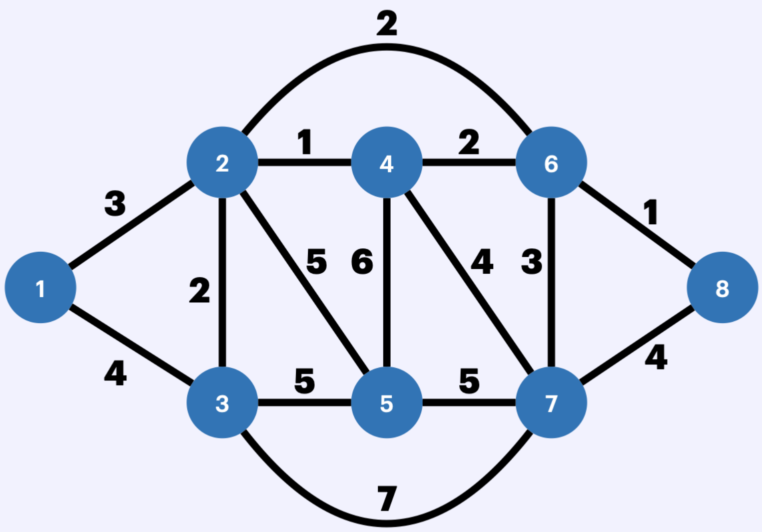
Exercise 5 Solution

By inspection, we find:



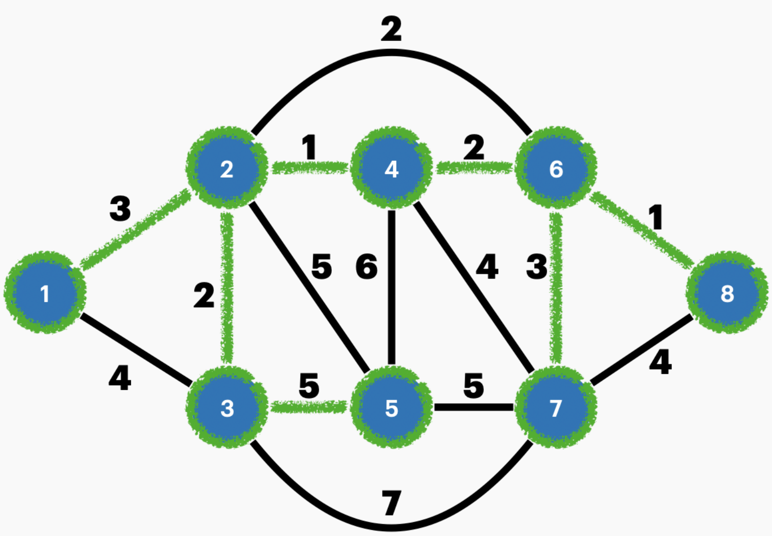
Exercise 5

Use Prim's Algorithm to find a different MST of the graph below:



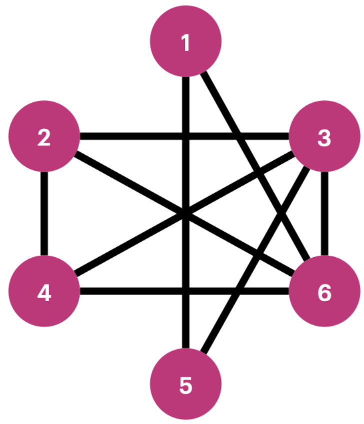
Exercise 5 Solution

According to Prim's Algorithm, we do the following: Starting from vertex 1, we add edges $\{1,2\}$, $\{2,4\}$, $\{2,3\}$, $\{4,6\}$, $\{6,8\}$, $\{6,7\}$ and finally $\{3,5\}$. This gives an MST of weight $3+1+2+2+1+3+5 = 17$. The MST is given visually below:



Problem Set Solutions

Graph Review



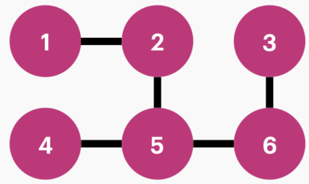
Graph A

1. For Graph A:
   (a) Find the neighbours of vertex 1 and vertex 6
   (b) Find a walk from vertex 1 to 4 and a path from 2 to 5
   (c) Find a cycle, a subgraph and a spanning tree
   (d) Find a 4-colouring

Solution:
(a) The neighbours of vertex 1 are 5 and 6. The neighbours of 6 are 1, 2, 3 and 4.
(b) Answers may vary. An example walk is 1, 5, 3, 2, 3, 4. An example path is 2, 3, 5.
(c) Answers may vary. An example cycle is 2, 3, 6, 4, 2. An example subgraph is the graph itself! An example spanning tree is the subgraph containing all vertices and edges $\{1, 5\}$, $\{5, 3\}$, $\{3, 2\}$, $\{2, 4\}$ and $\{4, 6\}$. This looks like a path!
(d) Answers my vary. A 4-colouring could be colour 1 and 2 as A, 3 as B, 4 and 5 at C and 6 as D.

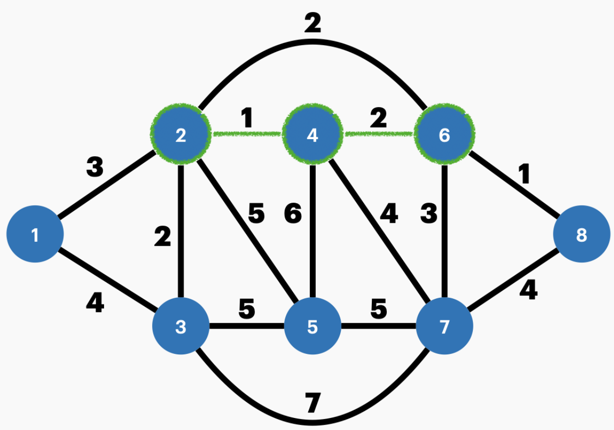
2. Draw a tree (the graph kind!)

Solution: Answers may vary. Here is an example:



3. In this problem we look at a weighted graph:
   (a) Draw a weighted graph with 8 vertices, labelled 1-8.
   (b) Find a path from vertex 2 to 6.
   (c) What is the weight of the path you found?

Solution: Answers may vary - be creative!

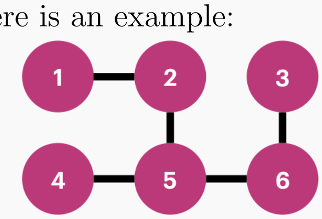


The weight of the highlighted path is $1 + 2 = 3$.

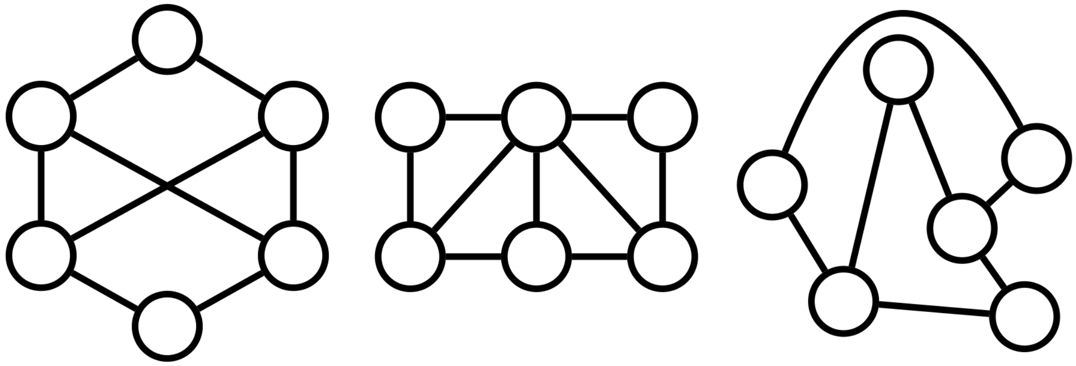
Graph Colouring

1. Draw a bipartite graph.

Solution: Answers may vary. Here is an example:



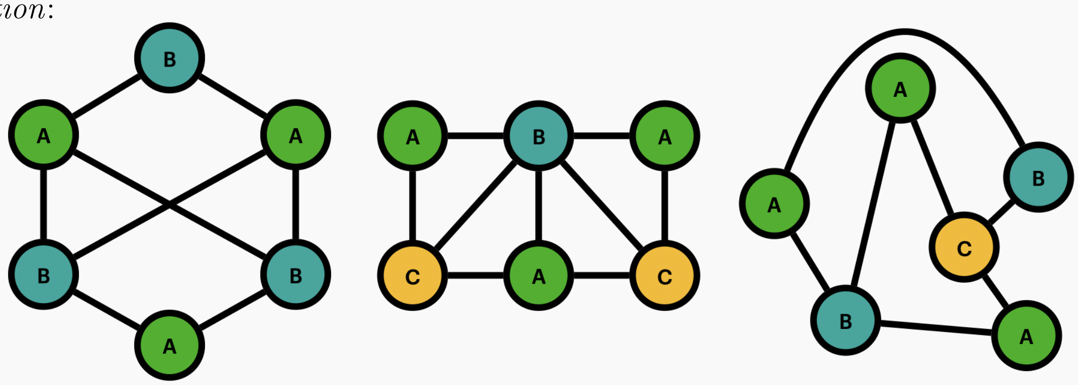
2. Which of the following graphs are bipartite?



Solution: Only the first graph is bipartite. The second and third graphs both have odd length cycles, whereas the first does not! See the solution to problem 3 for an example 2-colouring.

3. Find a colouring for each of the above graphs with the minimum number of colours.

Solution:



4. Find a condition that would make a graph **NOT**:

   (a) 3-colourable

   (b) 4-colourable

   (c) **Challenge:** k-colourable

Solution:

(a) A graph isn't 3-colourable if we have 4 vertices that are all connected to each other.

(b) A graph isn't 4-colourable if we have 5 vertices that are all connected to each other.

(c) A graph isn't k-colourable if we have $k+1$ vertices that are all connected to each other.

Timetabling

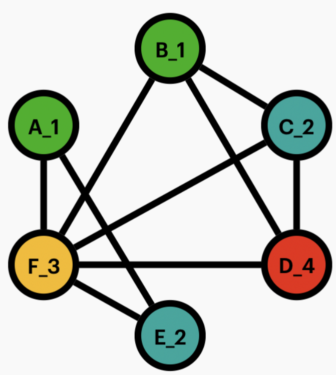
1. This problem is called "Aircraft Scheduling" and comes from the 2019 BCC. Use graph colouring to solve this problem:

   When an aircraft lands at an airport, it is assigned a designated airspace called a *corridor*. By ensuring that flights with similar landing times are in different corridors, air traffic controllers can help to avoid accidents.
   At the Bebrasland airport, two aircraft cannot have the same corridor if their landing times are within 15 minutes of each other.
   You are the Air Traffic Controller at the airport and your job is to assign corridors for the flights that are due to land at the times shown in the table.

Flight	Time
A	7:00am
B	7:21am
C	7:20am
D	7:18am
E	7:03am
F	7:12am

   What is the minimum number of corridors needed to ensure that the flights in the above table are assigned corridors according to the rules at the Bebrasland airport?

Solution: The minimum number is 4. Below is a graph with a colouring representing the situation. Here, we represent each of the flights as vertices (corresponding with the letters in the vertex labels) and each possible conflict as edges. Colours are represented by the numbers in the vertex labels.



2. **Challenge:** Think of a real-life example of where graph colouring might be used (like our example with aircraft or class scheduling). Create a practice problem for this example and

have a friend try solving the problem using graph colouring.

> *Solution*: Be creative! Examples could include scheduling for a retail store or planning when different items are produced in a factory.

Greedy Colouring Algorithm

1. Find a colouring for Graph B using the greedy colouring algorithm.

> *Solution*:
> Answers vary. Here is an example:
> Green - vertices 1, 4, 6
> Red - vertices 2, 7
> Yellow - vertices 3, 8
> Blue - vertex 5

2. Does your colouring use the minimum amount of colours? If not, see if you can find a colouring which uses the fewest colours possible.

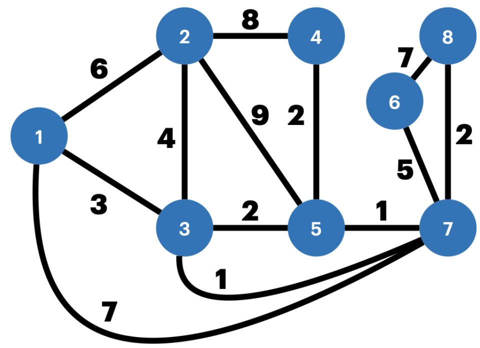
> *Solution*:
> Here is a colouring using the minimum amount of colours:
> Green - vertices 2, 7
> Orange - vertices 3, 4, 6
> Purple - vertices 1, 5, 8

3. **Challenge:** Create a graph, colour order and vertex order such that when the greedy colouring algorithm is applied (using the given orders), we do not achieve a minimum colouring.

> *Solution*:
> We can use Graph B with colour order Green, Red, Yellow, Blue and vertex order 1, 2, 3, 4, 5, 6, 7, 8.
> This is a pretty big graph. How simple of a graph can you make to satisfy this problem?

Prim's Algorithm

1. Find an MST of Graph C using Prim's Algorithm.



Graph C

Solution:
According to Prim's Algorithm, we do the following: Starting from vertex 1, we add edges $\{1,3\}$, $\{3,7\}$, $\{5,7\}$, $\{7,8\}$, $\{4,5\}$, $\{2,3\}$ and finally $\{6,7\}$. This gives an MST of weight 3+1+1+2+2+4+5 = 18. The MST is given visually below: